The concept of a function among prospective teachers

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Abstract: The study of functions is an important topic in the mathematics curriculum, relevant for the development of abstract thinking, the application of mathematics to everyday situations and the relationship with other domains of knowledge. However, often students do not have a satisfactory understanding of the function. This may be due to several factors, including the different conceptions of function that teachers themselves may have. As trainers of teachers, we investigate which notions of a function prospective teachers hold. The research conducted was qualitative, based on a questionnaire. The relevance of the topic, either from a mathematical or a didactical perspective, entails the need for designing new dynamics in teachers' education that are able to challenge intuitive conceptions and build proper connections.

Keywords: conceptions; function; prospective teachers

Introduction

The complexity of mathematical concept formation makes teacher training courses responsible for promoting identification dynamics and strategies in order to clarify prospective teachers' conceptions about key concepts in the development of mathematical knowledge. Interest in such conceptions is based on the assumption that there is a conceptual substratum that plays a determining role in thought and action (Ponte, 1992), which has relevance for the teaching of a given mathematical concept. Among the various concepts that organize math programmes, the one of a function is particularly notable for "providing a consistent way to make connections between and across a wide range of topics in mathematics itself and in other areas" (Santos & Barbosa, 2016, p. 144). It is a concept of an abstract nature in which many students manifest difficulties in their learning. These difficulties derive, according to Santos and Barbosa (2016), from the "diversity of ways of communicating it and (…) interpreting it, making it a fertile ground for studies of its teaching processes" (p. 114). The relevance of the concept of a function for the development of abstract thinking, and its relationship with other mathematical topics, justifies giving it a central concern in training prospective primary education teachers. In the realization of this objective, we analyse the notion of function in Portuguese school mathematics programmes and discuss the conceptions that prospective teachers hold about its role in mathematical practice.

The notion of a function in the Portuguese mathematics programmes

The Portuguese education system encompasses twelve years before higher education. The first nine years correspond to basic education and the last three to high school. Basic education consists of three cycles, lasting four, two and three years, respectively. The first cycle has a single teacher covering all subject areas. Along these three cycles of basic education, the math's curriculum is the same for all students. In the three years of high school the math's curriculum differs according to the specialization domain the student may take (Science, Humanities, Technology or Arts).

In their curricular organization, the mathematics programmes for the early years include topics that, in informal terms, express relationships somehow connected to the notion of a function. Examples include the result of arithmetic operations between two numbers or the relationship between a given geometrical figure and the measurement of its area. In teaching such topics, the relevance of mathematical communication in stressing language aspects that help students to understand the requirements underlying the notion of a function is revealed. This formally emerges in the 7th grade programme under the topic 'Functions, Sequences and Successions' (MEC, 2013). The topic deals in this grade with the identification of correspondences between sets that translate into functions. Moreover, it introduces the terminology associated with the identification of object, image, domain, target set, and range, as well as independent and dependent variables. This is followed by the representation of functions through arrow diagrams, tables and Cartesian graphs, as well as by a discussion of operations on functions, with the purpose of introducing the notions of linear function ($y = ax, a \neq 0$) and, subsequently, that of direct proportionality.

In the 8th grade, graphs of affine functions ($y = ax + b$) are studied through the identification of corresponding equations in the plane. This aims at making students aware that the graph of an affine function is obtained by translating the graph of a linear function with the same slope, taking into account the meaning of the parameters of the corresponding equation; it also introduces the procedure for obtaining the slope of this line knowing any two points in the domain. Finally, in the 9th grade, algebraic functions are studied through the definition of inverse proportionality, in their different representations (tabular, analytical and graphical), and the graphical interpretation of solutions of 2nd degree equations. This leads to the identification of the curves that represent functions of type $f(x) = ax^2$ (with $a \neq 0$) and the solution set of the 2nd degree equation $ax^2 + bx + c = 0$ (with $a \neq 0$) as the intersection of the equation for a parabola, $y = ax^2$, with the one for a line, $y = -bx - c$.

With the knowledge acquired during their training years at the 3rd cycle, it is intended that students become able to apply it when solving problems in various contexts. Later, in the transition to secondary education, this knowledge serves as a prerequisite for Grade 10, in which the curriculum covers defining the composition of functions and the inverse function of a bijective function, relating geometric properties of graphs to properties of the functions under study, identifying monotony ranges of real-variable real functions, identifying extremes of real-variable real functions, and studying elementary functions and algebraic operations on functions. In the 11th year the study of functions widens to include trigonometric functions, the Heine limits of real functions of a real variable, those derived from real functions of a real variable and their applications. Finally, in the 12th year, limits and continuity of real-variable functions and successions are studied, and the study of derivatives of real functions of a real variable is deepened.

Conceptions about functions

The education of mathematics teachers always assumes a number of epistemological, ideological and cultural positions with respect to teaching, the teacher and the students (Marcelo Garcia, 1999). However, at the level of initial education, the mathematical formation of prospective teachers is characterized by the teacher-knowledge-student relationships, contextualized by the way they observe, as students, the development of the mathematical curriculum (Ponte, 1992). Shulman (1986) points out the importance of didactic research in order to find answers to questions related to teaching content, teachers' knowledge of this content, where and when the latter is acquired, how and why it is transformed during formal training, and, finally, how it should be used in concrete classroom teaching. Cognitive in nature, conceptions are, on the one hand, fundamental in structuring the meaning assigned to reality and, on the other, important as a blocking element for new understandings (Ponte, 1992). The construction of a conception takes place in individuals (as a result of elaboration of their own experience) and at social level (from comparing their own synthesis with those of others). The formulation of mathematical concepts and the teaching/learning of mathematics are thus somewhat influenced by the dominant experiences and social representations. Therefore, the identification of conceptions in prospective teachers, in this case about the notion of function, provides indicators about their mathematical formation.

The applicability of mathematics, as a tool for studying real phenomena, depends on the conception of a given model that synthesizes and relates the main characteristics of the phenomenon to be handled. Such relationships are often represented by functions.

The concept of a function is the result of a long development of mathematical thinking (Caraça, 1984). Such historical evolution refers to the complexity of the construction of the notion of a function, and its acquisition as an abstract concept as a result of several mental constructions developed by mathematicians and scientists to solve problems and create theories (Evangelidou, Spyrou, Elia, & Gagatsis, 2004). This may explain some of the difficulties manifested by many students and prospective teachers in building up their own construction of this concept.

A study by Vinner and Dreyfus (1989) shows that college students during a course on calculus, even if they were able to give a correct explanation of the definition of function, did not apply the definition successfully. Breidenbach et al (1992) point-out that "college students, even those who have had a good number of math courses, do not have much understanding of the concept of function" (p. 247), confirming that it is a complex concept for students and its conceptual development requires a longer period of time than is typically assigned. Actually, the concept of a function must be dynamically introduced as a kind of relationship, correspondence, or covariation, rather than putting excessive emphasis on the static concept of (a set of) ordered pairs (Hansson, 2004). On the other hand, the diversity of representations associated with functions and the difficulty of establishing connections between them may become confusing on a first exposition. In practice, different teaching approaches to the concept of function arise from what students infer from vague information. Based on this assumption, Evangelidou et al (2004) conducted a study with prospective teachers, predominantly from primary school pre-service courses, seeking to understand the interpretation of the concept of a function among university students with respect to the conception itself, its use, and the identification of functions in multiple representations. The study revealed three strong trends in prospective teachers' notions about functions. The first is the identification of a function with the more specific concept of 'one-to-one function', common among a large percentage of students. Although this notion works for a wide range of situations involving functions, it becomes a strong obstacle to understanding the concept at a broader level. The second trend is the idea that a function is an analytical relationship between two variables. The third trend simply identifies functions with their representations, e.g. as a diagram or a Cartesian graph.

In the studies by Tall and Vinner, conceptions of the notion of a function are framed by the *concept image*, a mental construction that represents the cognitive structure associated with the concept, which includes all mental images and associated properties and processes, and the *concept definition*, i.e., the formal definition (Tall & Vinner, 1981;

Vinner & Dreyfus, 1989). Despite being introduced to the formal definitions of mathematical concepts, students tend not to use them when asked to identify or construct a concrete, related mathematical object. They often base their ideas on a conceptual image emerging from the set of all mental images associated in the student's mind with the name of the concept, along with all the properties that somehow may characterize them (Vinner & Dreyfus, 1989). Consequently, students' responses to concept-related tasks depend on these conceptions and deviate from expectations of institutional knowledge. Viirman, Attorps, and Tossavainen (2010) identified different categories of definitions and conceptual images concerning the notion of a function:

> *Correspondence/dependence relation.* A function is any match or dependency relationship between two sets that assigns to each element in the first set exactly one element in the other set.
> *Machine.* A function is a 'machine' or one or more operations that transform variables into new variables. In this case, no explicit mention of domain and range is made.
> *Rule/formula.* A function is a rule, a formula, or an algebraic expression.
> *Representation.* The function is identified with one of its representations.

For the authors, students' common images of the concept of a function have direct implications for teaching, as they can be used as a starting point for any prospective teaching of this concept.

Research methodology

In order to investigate the way a function is conceptualised among prospective primary education teachers, the paper focuses on four open questions: (1) What is the meaning given to the term function?; (2) (a) When do you use functions? (b) Who else uses functions and when?; (3) What mathematical symbol(s) do you use to represent functions?; (4) How would you explain the concept of function? The study included 87 prospective teachers who were organized into three groups according to the year they attended, as follows. Group A consisted of 29 students from the 1st year of the bachelor's degree in basic education (S1 to S29); Group B consisted of 40 students from the 3rd year of the bachelor's degree in basic education (S30 to S69); Group C consisted of 18 students from the 2nd year of the master's degree in teaching primary school and Mathematics and Natural Sciences at basic school (S70 to S87). The codification, from S1 to S87, followed the order of the students of each grade considered.

The data was subjected to content analysis and summarized around the following dimensions: meaning, use (when and who), symbolic representation, and concept. The information from data analysis, resulting from the answers given by the different groups of prospective teachers, is presented according to the following categories:

Question 1: Correspondence/ Dependency relationship; Machine; Formula/rule.
Question 2a: School context; Out-of-school context; School and out-of-school context.
Question 2b: Teachers and students; Specific professions; Any professional context.
Question 3: Isolated terminology; Analytical expression; Arrow diagram; Multiple representations.
Question 4: Common language; Algebraic expressions; Arrow diagram; Cartesian graph; Multiple representations.

Results

With respect to the question "What is the meaning given to the term function?", most students mentioned that it is a correspondence or dependency relationship, as shown in Table 1.

Table 1. Frequency of answers from different groups to question 1

Meaning of the term function	Group A	Group B	Group C	Total
Correspondence/ Dependency relationship	24	29	10	63
Machine	1	0	2	3
Formula / rule	0	1	1	2
Meaningless	3	9	5	17
No reply	1	1	0	2
Total	29	40	18	87

Although almost all students refer to the concept of a function as a relationship, it is possible to identify some differences between the responses. Some of them identify "a relationship between two sets" (S9), or stress it "relates to sets, images and objects" (S11), while others refer to a "relationship between variables" (S76). The notion of dependency can also be identified in some of the answers, for example, "function means a transformation of an x by a y. When we have a dependent variable and an independent variable" (S80) or "A function implies the existence of two variables (x, y). By organizing the regular data in a graph, we can predict the values of x or y, knowing other variables" (S55). A very limited number of participants, seven out of the 87, report that in a function, the relationship is single-valued. For example: "It is a function when an element of set A matches one and only one element of set B" (S21) or "A function is a mathematical concept. A function is when one element of the starting set corresponds to one and only one element of the target set" (S52).

Some answers adopt an operational perspective that can be classified as belonging to the category of Viirman et al (2000): that of a function regarded as a "machine". Examples include "A function is a mathematical method used to find an unknown value" (S23), "a function is something that allows us to determine y corresponding to an x or the opposite in a particular case" (S79), or "function means a transformation of an x by a y. When we have a dependent variable and an independent variable." (S80). Only two students associated the notion of function with a rule or formula, for example: "It consists of an expression with at least 2 unknowns, where one can verify the relationship between them" or "For me, a function is an expression that relates two variables, thus one being a dependent variable of another" (S76).

With respect to the second question, item (a), "When do you use functions?", the answers received were classified in the following categories: School context; Out-of school context; School and out-of-school context; Meaningless (Table 2).

Table 2. Frequency of answers from different groups to question 2.a

When using functions	Group A	Group B	Group C	Total
School context	11	29	11	51
Out-of-school context	9	4	4	17
School and out-of-school context	7	7	2	16
Meaningless	1	0	1	2
No reply	1	0	0	1
Total	29	40	18	87

The most frequently occurring category for each group was the school context. For example: "I only use functions when asked by teachers" (S47), "School or university" (S18), "In mathematics as a way of getting a relationship between two sets" (S3). Some students, on the other hand, argue that functions can be used in any context: "Functions are used when we need to establish relationships between two variables" (S55), "In mathematics classes and sometimes in everyday life" (S27).

For the second question, item b, the students were asked "Who else uses functions, and when?" The answers were organized according to the following categories: Academic context, Specific professions and Any professional context (Table 3).

Table 3. Frequency of answers from different groups to question 2.b

Who uses functions	Group A	Group B	Group C	Total
Academic context	7	13	4	24
Specific professions (engineers, nurses, etc.)	7	12	8	27
Any professional context	9	11	6	26
Meaningless	1	2	0	3
No reply	5	2	0	7
Total	29	40	18	87

Responses, while showing some diversity in terms of who uses functions, are less diverse in what concerns their use. For example, some answers point to the academic context: "Mathematicians and learners" (S1), "teachers in class" (S43), "Mathematicians and students in mathematics, as they are presented with a diverse number of problems where you have to decipher or apply functions" (S53).

Responses emphasizing other professional contexts are more diverse, as one would expect. For example: "Traders and their buyers" (S3), "Architects to carry out projects" (S21), "Who works with money and quantities" (S15), "Taxi drivers when calculating the total value of the fare; any seller who wants to know the full value of a purchase" (S30).

Some participants included in their answers academic and everyday contexts, for example, "Mathematicians mainly, but everybody uses them, on a daily basis, to solve mathematical problems, to calculate unknowns that arise in everyday life" (S31).

In the third question, participants were asked to indicate "What mathematical symbol(s) do you use to represent functions?"

Table 4. Frequency of answers from different groups to question 3

Symbols used to represent functions	Group A	Group B	Group C	Total
Isolated terminology	12	22	5	39
Analytical expression	11	11	8	30
Arrow diagram	2	4	0	6
Multiple representations	0	1	3	4
Meaningless	4	1	2	7
No reply	0	1	0	1
Total	29	40	18	87

Answers classified as "isolated terminology" are for example, "x, y, ( )" (S2); "A lowercase letter, ex. $f(6)$"; "F to represent a function and we have an image and an object" (S24). Several participants, on the other hand, resorted to analytical expressions, such as: "x and $f(x)$, for example, $f(x) = 2x^2$ or $g(x) = 2x + 1$" (S27); "$y = mx + a$; $y = x^2 + a$" (S53).

A small group used the symbolism of arrow diagrams, as, for example, the one shown in Figure 1.

(object) (image)

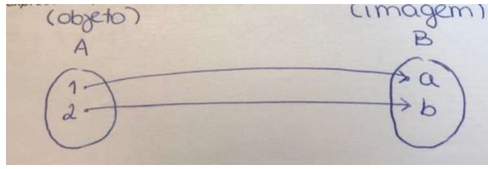


Figure 1. Reply (S22) to question 3

Some students presented more than one representation, as in the example shown in Figure 2. Such answers were classified as "multiple representations".

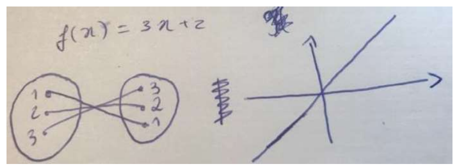


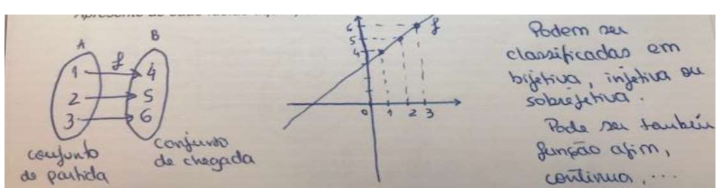
Figure 2. Reply (S80) to question 3

Finally, question 4 asked participants to explain "How would you explain the concept of a function?" The distribution of the answers per category is organized in Table 5.

Table 5. Frequency of answers from different groups to question 4

Explaining of the function concept	Group A	Group B	Group C	Total
Common language	3	3	4	10
Algebraic expressions	1	4	0	5
Arrow diagram	12	18	4	34
Cartesian graph	3	2	4	9
Multiple representations	5	4	5	14
Meaningless	5	9	1	15
Total	29	40	18	87

Some students tried to outline an explanation by resorting to common language, for example: "Something that is associated to something with a particular intent. For example, in 2 weeks I do 2 homeworks, in 4 weeks I do 4. Another example: My brother is 6 years old, I am 9. When he will be 18, I will be 21" (S2). This student provided two concrete examples. Actually, almost all students resorted to concrete examples, although represented in different ways. For example, Figure 3 shows a representation of the concept based on both an arrow diagram and a Cartesian graph. Furthermore, the student S20 adds a reference to some properties associated with the concept of a function: "they can be classified as bijective, injective or surjective"; and also "they can be affine, continuous, …".



(Starter set) (Target set)

Figure 3. Reply (S20) to question 4

Most students who resort to "multiple representations" apparently seek to capture situations involving both discrete and continuous sets, through the different approaches they have encountered through years of formal education. However, it is not always clear that this distinction is actually assumed by the students. For example, Student S20 depicted a line on the Cartesian graph suggesting a function the domain of which is a continuous set. However, the fact that the elements shown in the Cartesian graph and the arrow diagram are exactly the same may suggest he was just using different representations and that the line is mislabelled. Another example, complementing the one above, is the case of student S44 (Figure 4).

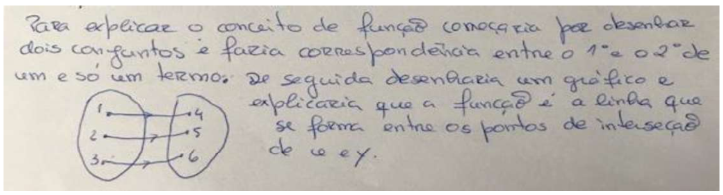


Figure 4. Reply (S44) to question 4

The student starts from an example presented through an arrow diagram with a discrete set and then states that "To explain the concept of a function, I would start by drawing two sets and match one to the other in terms of one and only one term. Then I would draw a graph and explain that the function is the line connecting the intersection points of x and y" (S44). This answer, like many others, reveals students' weakness in mastering the concept of a function. It should be added that the majority of answers to question 4 are based on examples only. Few students were concerned to provide additionally an explanation.

Conclusions and implications for mathematics education

The study reported in this paper provides a number of hints on how the notion of a function, central as it is in Mathematics, is conceptualised among a sample of students training to become teachers of mathematics. The participants express the concept of a function in terms of a relation between magnitudes, the existence of objects and images, and the construction of graphs, without, however, being precise on the nature of a functional correspondence. No relevant differences were found between the two groups of students under analysis. In general, the content of the answers given by the students attending the MSc degree does not reflect significant differences with respect to the ones given by the students attending the lower degree.

In general, functions are regarded by the participants in this study from a structural perspective, as some sort of set of pairs, rather than as a transformation or a rule, typically expressed as some form of algebraic formula. This is in contrast with findings in similar studies conducted within the same project in The Netherlands and Ireland (Oldham et al, 2019).

Although further comparative research is necessary to clearly identify the prevalence of different patterns in different national contexts, these differences may come from the different approaches to teaching this concept adopted in such contexts. Clearly, the way the study of functions develops throughout the curriculum contributes to shaping prospective intuitions. In Portugal, functions are initially studied in a numerical or tabular representation, to establish a single-valued relation between two sets consisting of few elements. As the first learning is the most resistant to change, this form of teaching may entail too static a perspective on the definition of a function (Hansson, 2004).

Similarly, the extended use of the arrow diagrams detected may be associated with the fact that most of the students considered have taken mathematics up to the 9th grade only, and it is usually through the use of arrow diagrams that the concept is introduced in basic education.

This piece of research also focuses on the way these students consider the applicability of the concept of a function. Actually, participants often stress, albeit in a rather concise way, the applicability of functions to everyday situations, for example to relate two variables in a study or to analyse data. Their technical use, however, may convey the message that the use of functions concerns mainly academics, teachers, engineers and researchers, rather than lay person in the street.

Further systematic research is required however. In any case, it seems urgent to design new dynamics in teachers' education able to put intuitive conceptions in question and build proper connections.

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